

Existence and Multiplicity of Solutions for a Nonvariational Elliptic Problem

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1. INTRODUCTION

There is a large literature on the existence and multiplicity of positive solutions for the problem

$$\Delta u + f(x, u) = 0 \quad (1.1)$$

$$u|_{\partial\Omega} = 0, \quad (1.2)$$

where $f(x, 0) > 0$, f is monotone increasing in u , and f satisfies some increasing rate in u uniformly in x (see for instance [10, 4, 3] and references there). In some cases [3] after establishing the existence of one solution, the variational technique of the Mountain Pass Lemma is utilized to establish the existence of a second solution. In others [4] degree theory and a priori estimates are employed to achieve the same result. It must be mentioned that each case usually requires different properties of f to obtain the desired results.

Here we consider the existence and multiplicity of positive solutions of the equation

$$\Delta u + \mathbf{b} \cdot \nabla u + \lambda f(x, u) = 0 \quad (1.3)$$

$$u|_{\partial\Omega} = 0. \quad (1.4)$$

It is clear that the problem does not admit any variational method of treatment; hence the Mountain Pass Lemma technique cannot be employed. Moreover, the a priori bound treatment in [4] also fails. We will introduce a treatment of the problem that establishes another a priori bound for Eqs. (1.3)–(1.4). Then employing Leray–Schauder degree theory, a second solution is established. This method can in fact be extended to a system of semilinear elliptic equations, and hence answers

one of the open questions posed in [10]. Such details will be presented in a forthcoming paper. In what follows, solution means classical solution in $C^2(\Omega) \cap C(\bar{\Omega})$.

Equations (1.3)–(1.4) are considered in a smooth bounded domain Ω in \mathbb{R}^n , $n \geq 2$. The following assumptions are made with respect to \mathbf{b} , f , and Ω :

(H1) $\mathbf{b} \in C^1(\bar{\Omega})$ and $f(x, 0) > 0$, $f_u(\mathbf{x}, u) > 0$, $f \in C^3(\bar{\Omega} \times \mathbb{R})$, and for sufficiently large u , there are constants $c_1, c_2 > 0$, independent of x , such that

$$c_1 u^s \leq f(x, u) \leq c_2 u^s,$$

where $1 < s < n/(n-2)$ for $n > 2$, and any $s > 1$ for $n = 2$.

(H2) Ω is convex, $\partial\Omega$ has positive curvature everywhere, and there exist $r, \delta > 0$ such that for all $u \geq 0$ and all $x \in \Omega$, $\equiv \{x \in \Omega \mid \text{dist}(x, \partial\Omega) < r\}$,

$$\nabla_x f(\cdot, u) \cdot \boldsymbol{\mu} \leq 0,$$

and

$$\mathbf{b} \cdot \boldsymbol{\mu} \geq 0,$$

where $\boldsymbol{\mu}$ is any unit vector satisfying

$$|\boldsymbol{\mu} - \mathbf{n}(x)| < \delta \tag{1.5}$$

and $\mathbf{n}(x)$ is defined for $x \in \Omega$ to be $\mathbf{n}(y)$, which is the unit outward normal vector at $y \in \partial\Omega$ with y defined by $|y - x| = \text{dist}(x, \partial\Omega)$ (the question of well definedness of $\mathbf{n}(x)$ is discussed in [4]).

Under these hypotheses, we shall prove that there exists a $\lambda^* > 0$ such that for $\lambda < \lambda^*$, there are at least two solutions, for $\lambda = \lambda^*$, there exists at least one solution, while for $\lambda > \lambda^*$, there is no solution. In case that strict convexity of f is assumed, i.e., $f_{uu} > 0$, then the statements can be made more precise for $\lambda = \lambda^*$: there exists exactly one solution for such λ , and it is a simple turning point.

We have the following application in mind when we study Eqs. (1.3)–(1.4). To model a gaseous reacting mixture, the fluid mechanics and the reactions involved among all the interacting species have to be described. The governing system of equations are given in Eqs. (55)–(58) in chapter 1 of [6]. They consist of the continuity and momentum equations, which are further supplemented by the equations of state for the gases. Moreover, the species and temperature evolution equations, which involve chemical reactions, are included. The reason for the coupling of the fluid mechanics and chemistry is because of the presence of temperature in the equation of state.

For a wide class of combustion phenomena known as deflagration, the Mach number of the flow is small. It is also found that most of the essential qualitative features are preserved under the constant density approximation [7, p. 25]. With such an assumption, the fluid mechanics are decoupled from the chemistry of the reacting mixture, and can be found a priori. If abundant amount of cold fresh mixture at temperature T_0 is fed into a combustion chamber, the reactant concentration will be roughly constant. With $u + T_0$ being the absolute temperature, the governing equations for the chemical reacting mixture will be reduced to Eqs. (1.3)–(1.4), which represent the temperature distribution at steady state. The term $\mathbf{b} \cdot \nabla u$ with \mathbf{b} under the restriction of assumption (H2) represents the convection into the combustion chamber, while the second derivative term is due to the heat conduction. The term $f(x, u)$ represents the heat source due to chemical reaction. Since the species concentrations are constant this will make f independent of x explicitly. The reaction rate is governed by Arrhenius's law (see [13, p. 16, 7]); i.e., $f = B(T_0 + u)^2 \exp(-E/R(u + T_0))$ for some positive constants B , R , and E . When the activation energy E is 0, f will be an increasing polynomial function of the temperature. Even when E is non-zero, still f behaves as in our assumptions for α in the appropriate range.

2. EXISTENCE OF SOLUTIONS

Clearly, to $\lambda = 0$ there corresponds the solution $u \equiv 0$. In this case the solution is unique. Let S denote the set of λ for which Eqs. (1.3)–(1.4) have a nonnegative solution. We shall begin by proving that S is an interval.

LEMMA 2.1. *If λ is sufficiently small then $\lambda \in S$.*

Proof. Let \bar{u} satisfy

$$\begin{aligned} \Delta \bar{u} + \mathbf{b} \cdot \nabla \bar{u} &= -1 \\ \bar{u}|_{\partial \Omega} &= 0. \end{aligned} \tag{2.1}$$

Then $\bar{u} > 0$ in Ω by the Maximum Principle. This is an upper solution of (1.3)–(1.4) for sufficiently small λ , since

$$\Delta \bar{u} + \mathbf{b} \cdot \nabla \bar{u} + \lambda f(x, \bar{u}) = -1 + \lambda f(x, \bar{u}) < 0.$$

The function $\underline{u} = 0$ is a lower solution and $\underline{u} < \bar{u}$. Thus a solution exists by standard monotone iteration techniques [13].

LEMMA 2.2. *If a solution of Eqs. (1.3)–(1.4) exists for $\tilde{\lambda} > 0$, then a solution exists for all λ such that $0 \leq \lambda \leq \tilde{\lambda}$.*

Proof. Let \tilde{u} satisfy Eqs. (1.3)–(1.4) with $\lambda = \tilde{\lambda}$. It is immediate that \tilde{u} is an upper solution for Eqs. (1.3)–(1.4) for any $\lambda < \tilde{\lambda}$. With the lower solution being $\underline{u} = 0$, a solution exists by monotone iteration.

It should be noted that such monotone iterations given minimal positive solution for each λ for which a solution exists. A minimal positive solution u of (1.3)–(1.4) satisfies $0 < u \leq v$ in Ω for all positive solutions v of (1.3)–(1.4). It is also clear from the monotone iteration technique that $u_{\min}^{\lambda_1} < u_{\min}^{\lambda_2}$ in Ω for $\lambda_1 < \lambda_2$, where $u_{\min}^{\lambda_i}$ is the minimal solution for $\lambda = \lambda_i$, $i = 1, 2$.

We will establish that there is a second solution besides this minimal positive solution.

LEMMA 2.3. *S is bounded.*

Proof. Let T be the inverse operator of $-\Delta - \mathbf{b} \cdot \nabla$ subject to zero Dirichlet boundary condition. The maximum Principle implies that T is a positive operator. T is also compact. Hence by the Krein–Rutman Theorem [14] there exists a positive eigenfunction φ and a positive eigenvalue μ_1 satisfying

$$\begin{aligned} \Delta \varphi + \mathbf{b} \cdot \nabla \varphi + \mu_1 \varphi &= 0 \\ \varphi|_{\partial\Omega} &= 0. \end{aligned} \quad (2.2)$$

We claim that the same is true for its adjoint operator: there exists a positive eigenfunction φ^* and the same positive eigenvalue μ_1 satisfying

$$\begin{aligned} \Delta \varphi^* - \mathbf{b} \cdot \nabla \varphi^* - \nabla \cdot \mathbf{b} \varphi^* + \mu_1 \varphi^* &= 0 \\ \varphi^*|_{\partial\Omega} &= 0. \end{aligned} \quad (2.3)$$

This can be seen by observing that for sufficiently large $\alpha > 0$ the Maximum Principle implies that the inverse operator of $-\Delta + \mathbf{b} \cdot \nabla + (\alpha + \nabla \cdot \mathbf{b})$ is positive. Hence by the Krein–Rutman Theorem, there exists a positive eigenfunction φ^* and a positive eigenvalue β such that

$$\begin{aligned} \Delta \varphi^* - \mathbf{b} \cdot \nabla \varphi^* - \nabla \cdot \mathbf{b} \varphi^* - \alpha \varphi^* + \beta \varphi^* &= 0 \\ \varphi^*|_{\partial\Omega} &= 0. \end{aligned}$$

Let $\mu = \beta - \alpha$. We have

$$\begin{aligned} (\mu_1 - \mu) \int_{\Omega} \varphi^* \varphi \, dx &= \int_{\Omega} (\Delta \varphi^* - \mathbf{b} \cdot \nabla \varphi^* - \nabla \cdot \mathbf{b} \varphi^*) \varphi \, dx + \int_{\Omega} \mu_1 \varphi \varphi^* \, dx \\ &= \int_{\Omega} (\Delta \varphi + \mathbf{b} \cdot \nabla \varphi + \mu_1 \varphi) \varphi^* \, dx = 0. \end{aligned}$$

Since $\int_{\Omega} \varphi \varphi^* dx > 0$, it follows that $\mu = \mu_1$ and φ^* satisfies

$$\begin{aligned} \Delta \varphi^* - \mathbf{b} \cdot \nabla \varphi^* - \nabla \cdot \mathbf{b} \varphi^* + \mu_1 \varphi^* &= 0 \\ \varphi^*|_{\partial\Omega} &= 0. \end{aligned} \quad (2.4)$$

Now multiply Eqs. (1.3)–(1.4) by φ^* and integrate over Ω . By hypothesis there exists $\alpha > 0$ such that for all $x \in \Omega$

$$f(x, u) > \alpha u. \quad (2.5)$$

Thus we get

$$\int_{\Omega} \mu_1 u \varphi^* dx = \lambda \int_{\Omega} f(x, u) \varphi^* dx \geq \lambda \alpha \int_{\Omega} u \varphi^* dx. \quad (2.6)$$

Therefore, for any $\lambda \in S$ we have

$$\lambda \leq \frac{\mu_1}{\alpha} < \infty. \quad (2.7)$$

Hence the lemma.

From the previous lemmas it follows that there exists a $\lambda^* = \sup S$ such that a solution exists for $\lambda \in [0, \lambda^*)$ or $\lambda \in [0, \lambda^*]$ and there is no solution for $\lambda > \lambda^*$.

3. A PRIORI BOUNDS FOR SOLUTIONS

In this section we will establish that for any solution u of (1.3)–(1.4) with $\lambda \geq \varepsilon > 0$, its $C^{2+\alpha}$ -norm is uniformly bounded in Ω .

LEMMA 3.1. *For $\lambda \geq \varepsilon > 0$ and any compact set $K \subset \Omega$, there exists a $C > 0$, which can depend on K , such that for all solutions of (1.3)–(1.4) (subject to the assumptions specified in the Introduction),*

$$\|u\|_{L^1(K)} \leq C. \quad (3.1)$$

Proof. For fixed λ choose $\alpha = (1 + \mu_1)/\lambda$. By the superlinearity of f there is $p > 0$, which depends on α , such that for all u , $f(x, u) > \alpha u - p$ on $\bar{\Omega}$.

As in Eq. (2.6), here we get

$$\int_{\Omega} \mu_1 \varphi^* u dx = \lambda \int_{\Omega} f(x, u) \varphi^* dx \geq \lambda \alpha \int_{\Omega} u \varphi^* dx - \lambda p \int_{\Omega} \varphi^* dx. \quad (3.2)$$

Then,

$$\int_{\Omega} u \varphi^* dx \leq \lambda p \int_{\Omega} \varphi^* dx.$$

Since $\lambda \leq \lambda^*$ and in any compact set $K \subset \Omega$, φ^* is bounded below by a constant $c > 0$ depending only on K . This leads to the uniform boundedness of $\|u\|_{L^1(K)}$ for any solution u of (1.3)–(1.4).

LEMMA 3.2. *There exists $\delta > 0$, that depends only on the geometry of Ω and $C > 0$ (which can depend on δ , besides other parameters) such that if we define*

$$\Omega_{\delta} = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) \leq \delta\}$$

then any solution u of (1.3)–(1.4) has the a priori bound

$$\|u\|_{L^{\infty}(\Omega_{\delta})} \leq C. \quad (3.3)$$

Proof. We shall use r and δ as defined in assumption (H2). Due to the assumed strict convexity of the domain Ω , together with $\nabla_x f(\cdot, u) \cdot \mathbf{n} \leq 0$, and $\mathbf{b} \cdot \mathbf{n} \geq 0$ for $x \in \Omega_r$, the conditions in Theorem 2.1' in [5] are satisfied. Thus any positive solution u is nondecreasing along the inward normal direction for at least a distance of r from $\partial\Omega$. Our additional assumptions on f and \mathbf{b} allow us to conclude from the same theorem that there exist $0 < r_0 \leq r$ and $0 < \delta_0 \leq \delta$, which depend on the geometry of Ω but are independent of the solution u , satisfying:

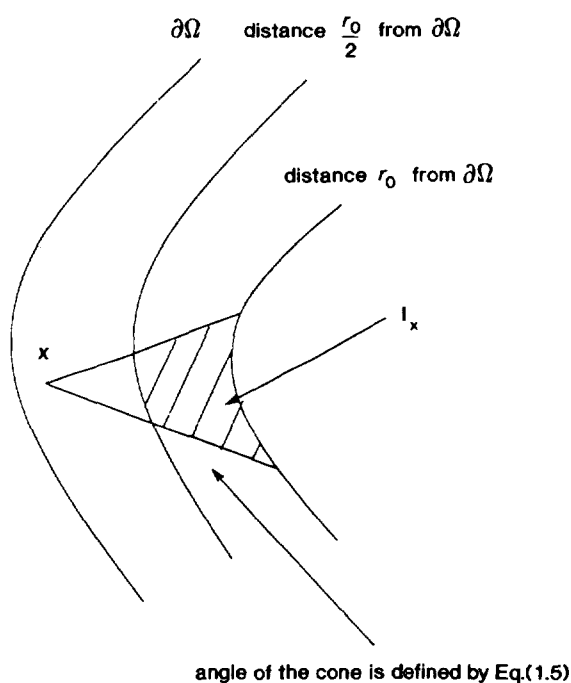
(*) For any $x \in \partial\Omega$, $u(x - t\mu)$ is nondecreasing for any $t \in [0, r_0]$, and μ such that $\delta_0 > |\mu - \mathbf{n}(x)|$.

Then the proof proceeds exactly as in Step 2 in Section 1 of [4]. We therefore will be brief in the following explanation, since the details can be found in the indicated reference.

The statement (*) implies that for $x \in \Omega_{r_0/2}$, there exists a truncated cone I_x as indicated in Fig. 1 such that $u(y) \geq u(x)$ for all $y \in I_x$. Moreover there exists a γ , independent of $x \in \Omega_{r_0/2}$, such that $\text{meas}(I_x) \geq \gamma > 0$ for all $x \in \Omega_{r_0/2}$. By Lemma 3.1,

$$C \geq \int_{I_x} u(y) \varphi^*(y) dy \geq \gamma u(x) \min_{y \in I_x} (\varphi^*(y)).$$

Since φ^* is positive in a compact set in Ω and r_0 is independent of u , so the last term on the right-hand side has a positive lower bound independent of u ; hence the proof.

FIG. 1. The truncated cone I_x .

From the previous two lemmas and since u is uniformly bounded near the boundary and locally in L^1 it follows immediately that

$$\|u\|_{L^1} \leq C_1. \quad (3.4)$$

Next we have

LEMMA 3.3. *For $\lambda \geq \varepsilon > 0$, there exists a $C_s > 0$ such that for all solutions of (1.3)–(1.4)*

$$\|u\|_{L^s} \leq C_s, \quad (3.5)$$

where s is the blow-up rate of f as defined in assumption (H1).

Proof. Let $\tilde{\varphi}$ be the solution of

$$\begin{aligned} \Delta \tilde{\varphi} - \mathbf{b} \cdot \nabla \tilde{\varphi} - \nabla \cdot \mathbf{b} \tilde{\varphi} &= -1 \\ \tilde{\varphi}|_{\partial\Omega} &= 0. \end{aligned} \quad (3.6)$$

We claim that $\tilde{\varphi} > 0$ in Ω . First, from (2.4) we have the existence of $\mu_1 > 0$ and $\varphi^* > 0$ in Ω such that $\Delta\varphi^* - \mathbf{b} \cdot \nabla\varphi^* - \nabla \cdot \mathbf{b}\varphi^* + \mu_1\varphi^* = 0$. By the continuous dependence of eigenvalues on the domain, we can construct a domain slightly larger than Ω so that the operator $\Delta\varphi^* - \mathbf{b} \cdot \nabla\varphi^* - \nabla \cdot \mathbf{b}\varphi^*$ has an eigenvalue μ no larger than and close to μ_1 . In particular $\mu > 0$ and its corresponding eigenfunction w is strictly positive in $\Omega \cup \partial\Omega$. We can then apply the Maximum Principle to $\tilde{\varphi}/w$ (see [11, Theorem 10 in Chap. 2]) and obtain that $\tilde{\varphi} > 0$ in Ω .

For every $\varphi \in H_0^2$, since $u \in C^2(\overline{\Omega})$, the divergence theorem gives

$$0 = \int_{\Omega} [\Delta u + \mathbf{b} \cdot \nabla u + \lambda f(x, u)] \varphi \, dx \quad (3.7)$$

$$= \int_{\Omega} [\Delta\varphi - \mathbf{b} \cdot \nabla\varphi - \nabla \cdot \mathbf{b}\varphi] u \, dx + \int_{\Omega} \lambda f(x, u) \varphi \, dx. \quad (3.8)$$

By the assumptions of f there is a sufficiently small $c_1 > c_0 > 0$ such that for all u , $f(x, u) > c_0 u^s$.

Thus with $\varphi = \tilde{\varphi}$ in (3.8),

$$\int_{\Omega} u \, dx = \lambda \int_{\Omega} f(x, u) \tilde{\varphi} \, dx \geq \varepsilon c_0 \int_{\Omega} |u|^s \tilde{\varphi} \, dx.$$

Hence using the result in (3.4), $u \in L_{\text{loc}}^s$. Together with Lemma 3.2 we conclude that there exists a constant C_s such that

$$\|u\|_{L^s} \leq C_s$$

for $\lambda \geq \varepsilon > 0$.

LEMMA 3.4. *For $\lambda \geq \varepsilon > 0$, there exists a $C > 0$ such that for all solutions of (1.3)–(1.4)*

$$\|u\|_{H^1} \leq C. \quad (3.9)$$

Proof. Multiply Eq. (1.3) by u , and perform integration by parts. By the assumptions on f we can find $c_3 > 0$ such that $f(x, u) < c_3(1 + |u|^s)$ uniformly for $x \in \Omega$. We get

$$\int_{\Omega} |\nabla u|^2 \, dx \leq c_3 \int_{\Omega} (\|\mathbf{b}\|_{\infty} |\nabla u| |u| + |u|^{s+1} + u) \, dx.$$

The first term on the right-hand side is bounded by

$$|\nabla u| |u| \leq \frac{\varepsilon_1}{4} |\nabla u|^2 + \frac{1}{\varepsilon_1} |u|^2, \quad \varepsilon_1 > 0.$$

Choose ε_1 so that $c_3(\varepsilon_1/4) \|\mathbf{b}\|_\infty = \frac{1}{2}$ to obtain

$$\int_\Omega \frac{1}{2} |\nabla u|^2 \leq c_3 \left(\int_\Omega \frac{\|\mathbf{b}\|_\infty}{\varepsilon_1} |u|^2 + |u|^{s+1} + u \right).$$

Since

$$\int_\Omega |u|^2 dx \leq |\Omega| + \int_\Omega |u|^{s+1} dx$$

and $\int_\Omega u dx$ is bounded by C_1 , we obtain that there is a constant c_4 , independent of u such that

$$\int_\Omega \frac{1}{2} |\nabla u|^2 \leq c_4 \left(\int_\Omega (|u|^{s+1} + 1) dx \right).$$

From the interpolation inequality

$$\|u\|_{L^p} \leq \|u\|_{L^q}^\eta \|u\|_{L^r}^{1-\eta}$$

with $1/p = \eta/q + (1-\eta)/r$, $0 \leq \eta \leq 1$, it follows that

$$\int_\Omega \frac{1}{2} |\nabla u|^2 \leq c_4 (\|u\|_{L^{n_c}}^\eta \|u\|_{L^s}^{1-\eta})^{s+1} + 1),$$

where $n_c \equiv 2n/(n-2)$, $\eta = n_c/(1+s)(n_c-s)$.

Since $\|u\|_{L^s}$ is bounded we have

$$\int_\Omega \frac{1}{2} |\nabla u|^2 \leq c_4 + c_5 \|u\|_{L^{n_c}}^\beta$$

with $\beta = \eta(s+1) < 2$ by assumption (H2).

Using Sobolev's estimate for $\|u\|_{L^{n_c}}$ we arrive at

$$\|u\|_{H^1}^2 \leq c_4 + c_5 \|u\|_{H^1}^\beta.$$

Hence the lemma.

LEMMA 3.5. *For $\lambda \geq \varepsilon > 0$, there exists a $C_0 > 0$ such that for all solutions of (1.3)–(1.4)*

$$\|u\|_{C^{2+s}} \leq C_0. \quad (3.10)$$

Proof. Since $\|u\|_{L^\gamma} \leq \|u\|_{H^1} \leq C$, the growth restriction on f implies [12, Prop. B.1, Appendix B] that $\|f(\cdot, u(\cdot))\|_{L^\gamma} \leq C$, with $\gamma > 2$.

By the regularity estimate for linear elliptic equations [1], $\|u\|_{W^{2,2}} \leq C$. We can now bootstrap, so that the function u is in C^α . Finally, using the Schauder estimate, $\|u\|_{C^{2+\alpha}} \leq C$ for $\lambda \geq \varepsilon > 0$.

An immediate consequence of (3.10) is the existence of a solution for $\lambda = \lambda^*$.

LEMMA 3.6. *There exists a minimal positive solution for $\lambda = \lambda^*$. Thus $S = [0, \lambda^*]$.*

Proof. Because of Lemma 3.5, we have a $C^{2+\alpha}$ bound on all solutions. Since the minimal solution for $\lambda < \lambda^*$ converges uniformly in $C(\bar{\Omega})$ to a function u^* as $\lambda \rightarrow \lambda^*$, a standard argument using the regularity of the elliptic equations shows that u^* is indeed a solution for $\lambda = \lambda^*$.

4. MULTIPLICITY OF SOLUTIONS

Let $\delta > 0$ such that $\delta\varphi$ is less than the minimal solution at $\lambda = \varepsilon/2$. Let X be the subspace of $C^{1+\alpha}(\bar{\Omega})$ whose elements vanish on $\partial\Omega$ and

$$G = \{u \in X \mid u \geq \delta\varphi, \|u\|_{C^{1+\alpha}} < C_0\}.$$

LEMMA 4.1. *Let $\varepsilon < \lambda \leq \lambda^* < \mu$. Then $\deg(I - \lambda T(f(x, \cdot)), G, 0) = 0$.*

Proof. From the previous section we know that uniform a priori bounds can be constructed for λ in the range of $\varepsilon \leq \lambda \leq \lambda^* < \mu$. As we homotopy from ε to μ , there cannot be a solution which touches $\delta\varphi$, since the minimal solution can only get larger with λ .

Since, in particular, there is no solution beyond λ^* , we have

$$0 = \deg(I - \mu T(f(x, \cdot)), G, 0) = \deg(I - \lambda T(f(x, \cdot)), G, 0).$$

LEMMA 4.2. *There exist at least two solutions of Eqs. (1.3)–(1.4), for λ in the range (ε, λ^*) .*

Proof. Let $\lambda \in [\varepsilon, \lambda^*)$ be fixed and pick λ_1, λ_2 such that $\varepsilon/2 < \lambda_1 < \lambda < \lambda_2 < \lambda^*$. Let \underline{u} be the minimal solution for λ_1 and \bar{u} the minimal solution for λ_2 . Then \underline{u}, \bar{u} are lower and upper solutions, respectively, for λ . Define

$$D = \{u \in C^{1+\alpha}(\bar{\Omega}) \mid \underline{u} < u < \bar{u}, \|u\|_{C^{1+\alpha}} < C_0\}.$$

Then with a proof similar to that of Lemma 3.4 in [9]

$$\deg(I - \lambda T(f(x, \cdot)), D, 0) = 1. \quad (4.1)$$

Thus by Lemma 4.1 and the excision property of the degree we conclude that

$$\deg(I - \lambda T(f(x, \cdot)), G \setminus \bar{D}, 0) = -1.$$

Hence there exists at least one more solution besides the minimal solution for $\varepsilon < \lambda < \lambda^*$.

5. BEHAVIOR OF THE SOLUTION FOR $\lambda = \lambda^*$ FOR STRICTLY CONVEX f

As in the case $\mathbf{b} = 0$ [3, 8], we can prove the following properties of the solution set if f is strictly convex, i.e., $f_{uu}(x, \cdot) > 0$ for all x in $\bar{\Omega}$:

(i) The solution for $\lambda = \lambda^*$ is unique.

(ii) Around a neighborhood of $\lambda = \lambda^*$, the solution set can be parametrized by $\lambda = \lambda(s)$ and $u = u(s)$ for $-\delta < s < \delta$ for some $\delta > 0$ with $\lambda(0) = \lambda^*$. Further, $\lambda(s) < \lambda^*$ for $s \neq 0$ in that neighborhood. Hence $\lambda = \lambda^*$ corresponds to a simple turning point.

To prove these claims, we define

$$F(u, \lambda) \equiv \Delta u + \mathbf{b} \cdot \nabla u + \lambda f(x, u) \quad (5.1)$$

for u in $C^2(\bar{\Omega})$ with zero Dirichlet boundary condition. Thus solutions of Eqs. (1.3)–(1.4) correspond to

$$F(u, \lambda) = 0. \quad (5.2)$$

Denote the minimal solution for λ by u_{\min}^λ ; hence the Fréchet derivative of F evaluated at the minimal solution is given by

$$F_u(u_{\min}^\lambda, \lambda)v \equiv \Delta v + \mathbf{b} \cdot \nabla v + \lambda f_u(x, u_{\min}^\lambda)v \quad (5.3)$$

for any v in $C^2(\bar{\Omega})$. For small λ the minimal solution is small in L^∞ norm (as can be demonstrated with similar arguments as in Lemma 2.1); hence the last term in the above equation is small and the Fréchet derivative is nonsingular by the Maximum Principle. Since $\lambda f_u(x, u_{\min}^\lambda)$ are increasing in λ , there is a first $\lambda = \tilde{\lambda} \leq \lambda^*$ such that F_u becomes singular. This is because otherwise we can continue the minimal solution branch by the implicit function theorem for all $\lambda \leq \lambda^*$ since the solution can never blowup due to the a priori bound that we have established. This will give a solution to (1.3)–(1.4) with $\lambda > \lambda^*$, which is a contradiction.

For $\lambda = \bar{\lambda}$, we denote the corresponding minimal solution by u_0 , i.e., $u_0 \equiv u_{\min}^{\bar{\lambda}}$. By an argument similar to those in earlier sections, there exists a positive (first) eigenfunction ϕ satisfying Eq. (5.3) and zero Dirichlet boundary condition, and a positive eigenfunction ϕ^* satisfying

$$\begin{aligned} \Delta \phi^* - \mathbf{b} \cdot \nabla \phi^* - \nabla \cdot \mathbf{b} \phi^* + \bar{\lambda} f_u(x, u_0) \phi^* &= 0 \\ \phi^*|_{\partial\Omega} &= 0. \end{aligned} \quad (5.4)$$

It can be checked that $F_{\bar{\lambda}}(u_0, \bar{\lambda}) = f(x, u_0) > 0$. Since $\int_{\Omega} \phi^* f(x, u_0) > 0$, $F_{\bar{\lambda}} \notin \text{range}\{F_u\}$ for $\lambda = \bar{\lambda}$. So $\bar{\lambda}$ is not a bifurcation point [2]. We can therefore parametrize the solution set in a neighborhood around $\bar{\lambda}$ by: $u = u(s)$, $\lambda = \lambda(s)$, for some sufficient small $\delta > 0$ and $-\delta < s < \delta$ with $\lambda(0) = \bar{\lambda}$ as a consequence of the implicit function theorem.

With the assumed smoothness in f , we can differentiate Eq. (5.2) with respect to s , which gives

$$\Delta v + \mathbf{b} \cdot \nabla v + \lambda f_u(x, u)v + \lambda'(s)f(x, u) = 0, \quad (5.5)$$

where $v \equiv (\partial u / \partial s)(s)$. We evaluate the equation at $s = 0$, multiply by ϕ^* , and integrate over Ω , which results in

$$\lambda'(0) = 0. \quad (5.6)$$

Differentiate Eq. (5.5) once more. Evaluating again gives

$$\bar{\lambda} \int_{\Omega} f_{uu} v(0)^2 \phi^* = -\lambda''(0) \int_{\Omega} f \phi^*. \quad (5.7)$$

By our assumption of convexness of f , $\lambda''(0) < 0$. Thus around a neighborhood of λ^*

$$\lambda = \bar{\lambda} + \lambda''(0)s^2 + O(s^3) \quad (5.8)$$

because of (5.6). So $\bar{\lambda}$ is a simple turning point.

Finally we show that $\bar{\lambda} = \lambda^*$, and there is only one solution for $\lambda = \lambda^*$. This will finish the proof of our claims.

Assume u_0 is the minimal solution at $\bar{\lambda}$ and u_* is the minimal solution at λ^* satisfying Eq. (5.2). Subtract one equation from another, multiply by ϕ^* , and integrate. This yields

$$\int_{\Omega} (\bar{\lambda} f(x, u_0) - \lambda^* f(x, u_*) - \bar{\lambda} f_u(x, u_0)(u_0 - u_*)) \phi^* = 0,$$

which can be written as

$$\begin{aligned} & \int_{\Omega} \bar{\lambda}(f(x, u_0) - f(x, u_*) + f_u(x, u_0)(u_* - u_0))\phi^* \\ &= \int_{\Omega} (\lambda^* - \bar{\lambda}) f(x, u_*)\phi^*. \end{aligned}$$

By the convexness assumption on f , the left-hand side is negative unless $u_* = u_0$ when it is zero. The right-hand side is nonnegative since $\lambda^* \geq \bar{\lambda}$, and can only be zero when $\lambda^* = \bar{\lambda}$. Hence the above equation holds only when $\lambda^* = \bar{\lambda}$, and $u_0 = u_*$.

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